Modeling and analysis of shape memory alloy wire

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Outline

- Introduction and experiments
- Modeling and analysis (3D and 1D)
Shape memory alloys (SMAs)

Examples: Ni-Ti, Cu-Zn, Cu-Al-Ni etc.

Characteristics:
- lighter weight, biocompatibility;
- excellent engineering functionality (e.g. sensing and actuation);
- coupling of mechanical and thermal fields

Two unusual properties
- shape memory effect:
  It can “remember” its original shape and recover to the undeformed shape when temperature is increased.
- superelasticity/pseudoelasticity:
  It can undergo large strain (upto 8%) without permanent damage upon loading and then revert to their original shape when force is removed.
Applications

(a) Ni-Ti stent  
(b) Ni-Ti braces  
(c) frame

(d) robot hand  
(e) sensor  
(f) valve

Figure 1: Applications
Mechanism

- **Two phases**: 
  - **Austenite** phase \( A \), face-centered cubic structure 
  - **Martensite** phase \( M \), body-centered tetragonal structure

- **Phase transition** between \( A \) and \( M \) (or variants) induced by temperature or by stress

- **Two properties**: shape memory effect and **pseudoelasticity**

Figure 2: illustration of phases and properties

(a) two phases  
(b) pseudoelasticity
Experiments on pseudoelasticity

1.3 Experimental studies on PE

\[ \varepsilon_m = 8\% \]

(a) nucleation
(b) propagation

Figure 1.4: (a) Stress-strain curves with a range of maximum strains from [94], (b) the specific stress-strain curve with maximum strain 8% from [94].

In the experiment, the ambient temperature is 353 K which is above \( A_f = 323 \) K and the wire is subject to uniaxial tension. Different curves in the figure 1.4(a) are associated with different maximum total strain before unloading, Figure 1.4(b) is the specific stress-strain curve with maximum strain 8%. Some unusual features are observed in Figure 1.4(b). Initially the wire is in austenite phase and undergoes linear elastic deformations. Once the stress reaches the peak stress i.e. martensite nucleation stress, the forward phase transition from \( A \) to \( M_d \) takes place and a small martensite band immediately starts to form. The formation of band indicates localized inhomogeneous deformations, rather than homogeneous deformations with homogeneous phase state. Meanwhile, the stress will drop to and stay at the stress plateau i.e. martensite propagation stress, along which the nucleated martensite band will grow by the propagation of transformation front until the wire is fully transformed to \( M_d \). Further loading will lead to the elastic deformation in martensite phase, the stress starts to increase again. Similarly in the unloading process, after the initial elastic deformation in martensite phase, the reverse phase transition from \( M_d \) to \( A \) will take place at the stress valley i.e. austenite nucleation stress. Then the austenite band will grow by the propagation of transformation.

Figure 3: Stress-strain curves of a wire under tension

(a) Tobushi et al. 1993
(b) Tse and Sun 2000
Experiments on pseudoelasticity

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Instability at nucleation: transition from one solution to another
Modeling and analysis

- Deformation of a solid material
- 3D energy and equations
- From 3D to 1D
- 1D analysis: phase plane analysis/analytical solutions
Deformation

In Cartesian coordinates

- reference state $\mathbf{X} = (X_1, X_2, X_3)^T$
- deformed state $\mathbf{x} = (x_1, x_2, x_3)^T$
- displacement $\mathbf{u} = \mathbf{x} - \mathbf{X}$

- deformation gradient

$$F = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}, \quad F_{ij} = \frac{\partial x_i}{\partial X_j}$$

- energy $\Phi(F)$ is function of $F$ or strain $F - I$. 
We consider a slender SMA circular cylinder with length $L$ and radius $a$.

Cylindrical coordinates from $X = (R, \Theta, Z)$ to $x = (r, \theta, z)$.

Consider symmetric deformation $\theta = \Theta$, and the other displacements are

$$U(R, Z) = r - R, \quad W(R, Z) = z - Z,$$  \hspace{1cm} (2)
The 3-D system

The stress tensor/matrix is defined by

\[ S_{ji} = \frac{\partial \Phi(F)}{\partial F_{ij}}, \quad i, j = 1, 2, 3. \]  (3)

PDE system

Two equations for \( U, W \)

\[ \text{Div } S = 0, \]  (4)

Boundary conditions

\[ S_{Rr} \big|_{R=a} = 0, \quad S_{Rz} \big|_{R=a} = 0, \]  (5)
\[ S_{Zz} \big|_{Z=0,L} = \gamma, \quad S_{Zr} \big|_{Z=0,L} = 0, \]  (6)
In general, energy $\Phi(F)$ is convex; like elastic energy.

For isotropic material, $\Phi$ can be determined by a series of material parameters up to certain order of approximation. For example, energy of a spring (Hooke’s law)

$$\Phi = \frac{E}{2} (\partial_Z W)^2. \quad (7)$$

To model phase transitions we assume $\Phi(F)$ is non-convex; reflected by material parameters.

In this problem, we have $\Phi(\partial_Z W, \partial_R W, \partial_Z U, \partial_R U)$
From 3D to 1D

Mathematical techniques

- Scaling or dimensionalization
  \[ \tilde{Z} = \frac{Z}{L}, \quad \tilde{R} = \frac{R}{L}, \quad \tilde{W} = \frac{W}{H}, \quad \tilde{U} = \frac{U}{H\tilde{R}}, \quad \epsilon = \frac{H}{L}, \]  
  where \( \epsilon \) is a small parameter and \( \tilde{R} \) is a small variable.

- Asymptotic expansions
  \[ \Phi(\tilde{W}, \tilde{U}; \epsilon) = \Phi_0(\tilde{W}, \tilde{U}) + \epsilon \Phi_1(\tilde{W}, \tilde{U}) + \epsilon^2 \Phi_2(\tilde{W}, \tilde{U}) + \epsilon^3 \Phi_3(\tilde{W}, \tilde{U}) + \ldots \]  
  (9)

- Taylor series expansion
  \[ \tilde{W}(\tilde{R}, \tilde{Z}) = \tilde{W}_0(\tilde{Z}) + \tilde{R}\tilde{W}_1(\tilde{Z}) + \tilde{R}^2\tilde{W}_2(\tilde{Z}) + \ldots \]  
  (10)

From 3D to 1D: substituting to the PDE system and make suitable truncations. Mathematical software is helpful.
The 1D model

- 1D equation

\[
V + D_1 V^2 + D_2 V^3 + a^2 \left(-\frac{1}{4} V'' + D_3 (V')^2 + 2D_3 VV''\right) = \gamma \tag{11}
\]

where \( V(\tilde{Z}) = \tilde{W}_0'(\tilde{Z}) \) is the axial strain.

- Free-end boundary condition (uniform deformation)

\[
V'(0) = V'(1) = 0 \tag{12}
\]

- For non-convex energy, there are requirements on parameters \( D_1, D_2, \) and we choose

\[
D_1 = -18, \quad D_2 = 100, \quad a = 0.04, \quad D_3 = -5. \tag{13}
\]

Homework: what conditions are needed for up-down-up curve of \( f(V) = V + D_1 V^2 + D_2 V^3 \)?
How to solve

\[ V + D_1 V^2 + D_2 V^3 + a^2 \left( -\frac{1}{4} V'' + D_3 (V')^2 + 2D_3 VV'' \right) = \gamma \]  \hspace{1cm} (14)

For each parameter \( \gamma \) (the force), how many solutions are there? Analytical expression or numerical solution?
First-order system

- Reduce the order

\[ V' = Y \]
\[ Y' = \frac{V + D_1 V^2 + D_2 V^3 - \gamma + a^2 D_3 Y^2}{a^2 \left( \frac{1}{4} - 2D_3 V \right)} \]  (15)

- Like a dynamical system? phase plane analysis

- Seek solutions with conditions

\[ Y(0) = Y(1) = 0. \]  (16)
One case

Using pplane8 in matlab, with $\gamma = 0.0168$. (Homework : stability of three equilibria)

\[ x' = y \]
\[ y' = (x + d_1 x^2 + d_2 x^3 - g + a^2 d_3 y^2)/(a^2 (1/4 - 2 d_3 x)) \]

- $a = 0.04$
- $g = 0.0168$
- $d_2 = 100$
- $d_3 = -5$
- $d_1 = -18$

The second unstable trajectory left the computation window. Ready.

The first unstable trajectory --> a nearly closed orbit.

The second unstable trajectory left the computation window. Ready.
Recall 1D equation

\[ V + D_1 V^2 + D_2 V^3 + a^2 \left( -\frac{1}{4} V'' + D_3 (V')^2 + 2D_3 V V'' \right) = \gamma \] (17)

Find first integral (Homework).
Analytical solution

- Recall 1D equation

\[ V + D_1 V^2 + D_2 V^3 + a^2 \left( -\frac{1}{4} V'' + D_3 (V')^2 + 2D_3 V V'' \right) = \gamma \]  

(17)

Find first integral (Homework).

- Example

\[ V + D_1 V^2 - a^2 \frac{1}{4} V'' = \gamma \]  

(18)

multiply \( V' \) and integrate

\[ \frac{1}{2} V^2 + \frac{D_1}{3} V^3 - a^2 \frac{1}{2} (V')^2 = \gamma V + C_0, \]

(19)

\[ (V')^2 = G(V) := a^2 (V^2 + \frac{2D_1}{3} V^3 - 2\gamma V + C_0) \]

Then

\[ Z = \int_{V_0}^{V} \frac{1}{\sqrt{G(s)}} \, ds \]  

(20)
Analytical solution

Suppose we get (Elliptic function)

\[(V')^2 = G(V; \gamma, C_0),\quad Z = \int_{V_0}^{V_1} \frac{1}{\sqrt{G(s)}} ds\]  \hspace{1cm} (21)

Given \(\gamma\), boundary conditions

\[V'(0) = V'(1) = 0,\]  \hspace{1cm} (22)

imply

\[\int_{V_0}^{V_1} \frac{1}{G(s)} ds = \frac{1}{n},\]  \hspace{1cm} (23)

where \(V_0, V_1\) are two roots of \(G\) and each integer \(n\) determines a \(C_0\) and hence a nontrivial solution.

- nontrivial solution corresponds to solution after phase transition.
- compare energy to select optimal solution.

Thank you for your attention!

Questions?